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The vorticity jump across a shock in a non-uniform flow

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The vorticity jump across an unsteady curved shock propagating into a two-dimensional non-uniform flow is considered in detail. The exact general expression for the vorticity jump across a shock is derived from the gasdynamics equations. This general expression is then simplified by writing it entirely in terms of the Mach number of the shock M_S and the local Mach number of the flow ahead of the shock M_U.

The vorticity jump is very large at places where the curvature of the shock is very large, even in the case of weak shocks. Vortex sheets form behind shock-shocks (associated with kinks in the shock front).

The ratio of vorticity production by shock curvature to vorticity production by baroclinic effects is $O(\frac{1}{2}(\gamma-1)M_U^2)$, where γ is ratio of specific heats, which is very small if the flow ahead of the shock is only weakly compressible. If, however, the tangential gradient along the shock of M_U^2 is large then baroclinic production is significant; this is the case in turbulent flows with large gradients of turbulent kinetic energy $\frac{1}{2}M_U^2$. The vorticity jump across a weak shock decreases in proportion to shock intensity if the flow ahead of the shock is rotational, rather than in proportion to the cube of shock intensity as is often assumed, and thus is not negligible. It is also shown that vorticity may be generated across a straight shock even if the flow ahead of the shock is irrotational. The importance of the contribution to the vorticity jump by non-uniformities in the flow ahead of the shock has not been recognized in the past.

Examples are given of the vorticity jump across strong and weak shocks in a variety of flows exhibiting some properties of turbulence.

1. Introduction

It is important to understand the way vorticity is generated across a shock front because vorticity is central to a wide variety of processes and phenomena. Turbulence is fundamentally vortical, and any changes in turbulence vorticity across a shock affects greatly the subsequent evolution of the turbulence behind the shock. For example, Lee, Lele & Moin (1993) found that the variance of the vorticity behind the shock is the main contributor to turbulent kinetic energy (TKE) dissipation. A curved shock generates vorticity unevenly and hence increases the variance of vorticity behind the shock. Eddy shocklets form spontaneously in flows with a turbulence Mach number $M_t = (\overline{u'_j u'_j}/a^2)^{1/2} > 0.3$, where u'_j are the turbulent velocity components, the overbar means ensemble average, and a is the speed of sound (Kida & Orszag 1990); thus the effects of vorticity generation across a shock are especially important in compressible turbulent flows.

Conversely, an initially laminar flow can be made turbulent if enough vorticity is added as the flow passes through a curved shock. The vortex sheets behind a shock with regions of high curvature become unstable and eventually produce three-dimensional turbulent motion in a similar way to that by which the large eddies of a mixing layer produce three-dimensional turbulence.

Vorticity is also a major factor in determining the combustion rate of a burning fluid; high vorticity tends to mix the fluid more efficiently and enhance the combustion rate. The fact that the generation of vorticity across a shock front and across a general flow discontinuity is very similar (Berndt 1966) means that many of the results presented here should also hold for flame fronts.

An expression for the vorticity jump across a shock wave was first obtained by Truesdell (1952). He considered the vorticity jump in a tangential direction $\boldsymbol{b} = \boldsymbol{n} \times \boldsymbol{s}$ (where $\boldsymbol{n}$ and $\boldsymbol{s}$ are respectively the unit normal and tangent vectors to the shock surface) across a two-dimensional steady shock in uniform flow, and found the relation

$$\delta\omega = -\frac{\mu^2}{1+\mu} U_S K, \tag{1.1}$$

where the symbol δ indicates the jump of a quantity across the shock,

$$\mu = \frac{\rho_b}{\rho_a} - 1 \tag{1.2}$$

is the normalized jump in density ρ across the shock (the subscripts a and b indicate quantities ahead of and behind the shock respectively), U_S is the velocity tangential to the shock in the frame of reference of the shock, and K is the curvature of the shock.

Truesdell's derivation uses Crocco's law relating the vorticity to the entropy gradient

$$\boldsymbol{u} \times \boldsymbol{\omega} = T \nabla S \tag{1.3}$$

(where $\boldsymbol{u}$ is the velocity vector, $\boldsymbol{\omega}$ is the vorticity vector, T is the temperature and S is the entropy) and hence his derivation involves the equation of state of the gas. The application of Crocco's law requires a number of assumptions: the flow must be steady, isocompositional and isoenergetic. Truesdell found, however, that the magnitude of the vorticity jump across a shock of a given strength and curvature depends only on the magnitude of the tangential component of the velocity and the curvature of the shock and is independent of the form of the equation of state.

Truesdell's (1952) result was later re-derived independently and generalized to the case of three-dimensional shocks by Lighthill (1957). Lighthill expressed Truesdell's vorticity jump relation in terms of the axes of principal curvature of the shock. Lighthill considered strong shocks, but that restriction was not necessary.

Hayes (1957) noticed that Truesdell's final relation does not depend on the thermodynamic state of the fluid, which suggests that the vorticity jump is a purely dynamical process. Following this idea Hayes was able to derive the vorticity jump using only conservation of momentum (the Euler equations) and conservation of mass. Hayes first derived Truesdell's result (extended to three-dimensional shocks) and then generalized it from a steady shock in uniform flow to the general case of an unsteady shock moving into a non-uniform flow. Hayes's dynamical derivation has the advantage that is does not require the restrictive assumptions of Crocco's law.

The general expression obtained by Hayes for the vorticity jump in the tangential direction $\boldsymbol{b}$ across an unsteady three-dimensional shock moving into a non-uniform flow is

$$\delta\omega\,\boldsymbol{b} = \boldsymbol{n} \times \left[-\frac{\partial(\rho C_r)}{\partial S}\delta(\rho^{-1}) + (\rho C_r)^{-1}(\mathrm{D}_S \boldsymbol{U}_S + C_r \mathrm{D}_S \boldsymbol{n})\delta(\rho) \right], \tag{1.4}$$

where $\partial/\partial S$ is the tangential part of the directional derivative, $C_r = C - A$ is the shock speed relative to the normal component of the flow ahead of the shock A, and D_S is the tangential part of the total time derivative. Note that the normal component of the vorticity is continuous across a shock. In evaluating equation (1.4) to obtain his final expression for the vorticity jump

$$\delta\omega\,\boldsymbol{b} = -\frac{\mu^2}{1+\mu}\boldsymbol{n} \times \left(\boldsymbol{U}_S \boldsymbol{\cdot} \boldsymbol{K} + \frac{\partial C_r}{\partial S} \right) \boldsymbol{s}, \tag{1.5}$$

Hayes assumed that the flow ahead of the shock is *uniform* (this expression contains a typographical sign error in Hayes 1957).

Berndt (1966) found an even more general expression for the jump in vorticity across a discontinuity surface which is not a contact surface. The vorticity jump is derived using two reference frames normal to the shock, one on each side of the discontinuity surface, chosen so that the tangential component of velocity vanishes in each frame. Berndt derived his most general result (which allows arbitrary motion of the fluid and the discontinuity) entirely kinematically and then included the dynamics of the flow by enforcing successively the momentum equation of the flow and the conditions of conservation of mass and momentum across the discontinuity. Berndt's most general dynamical expression includes the effect of an extraneous force field, a normal impulsive force and a discontinuity in the tangential velocity. Hayes's expression (1.4) is a special case of Berndt's general dynamical result. The general kinematical expression for the vorticity jump obtained by Berndt is

$$\delta\boldsymbol{\omega}\,\boldsymbol{b} = \boldsymbol{n} \times \left[\delta(D_S \boldsymbol{u}/u_n) - \nabla(\delta v_n) \right] + \delta\boldsymbol{\omega}_n + 2\,\delta\boldsymbol{\Omega}_S, \tag{1.6}$$

where $\boldsymbol{v}$ is the flow velocity relative to a common frame and $\boldsymbol{\Omega}$ is the angular velocity of the local frame with respect to some common frame.

The purpose of the present paper is to derive a general expression for the vorticity jump across a shock using the methods introduced in Kevlahan (1996) and then to use this expression to investigate mechanisms for vorticity production in the shock–turbulence interaction. The 'raw' equation for the vorticity jump will be equivalent to Hayes' result (1.4), but in evaluating this expression the flow ahead of the shock will be allowed to be non-uniform which results in several additional terms compared with equation (1.5). By writing the vorticity jump in a variety of special cases (e.g. strong shock, weak shock, strong flow ahead of the shock) a number of interesting facts may be discovered. The vorticity jump across a weakly curved strong shock can be calculated analytically and examples are given for a variety of flows ahead of the shock. The vorticity jumps in some of the unsteady weak shock interactions considered in Kevlahan (1996) are also found.

A central theme of this paper is the role played by non-uniformities in the flow ahead of the shock in generating vorticity across a shock; this mechanism is almost always neglected in discussions of vorticity generation across a shock.

2. Derivation of the vorticity jump across a shock

2.1. *The general equation*

In this section we derive the general expression for the vorticity jump across a two-dimensional unsteady shock in non-uniform flow, and evaluate this expression to the same level of generality.

The vorticity ω_b just behind a shock is

$$\omega_b = \frac{\partial v_b}{\partial x} - \frac{\partial u_b}{\partial y}, \tag{2.1}$$

where u and v are respectively the x- and y-components of velocity. We have taken the vorticity in the direction $-\boldsymbol{b} = -(\boldsymbol{n} \times \boldsymbol{s})$ so that positive vorticity is in the direction out of the plane of the flow (i.e. in the conventional z-direction). Unfortunately, the Rankine–Hugoniot jump conditions giving u_b and v_b cannot be used to calculate vorticity behind the shock ω_b because the curl operator involves exterior derivatives (derivatives in the direction normal to the shock front). The problem is similar to that encountered in Kevlahan (1996) when calculating the evolution of shock strength μ and its normal derivatives. We follow a similar procedure here to calculate the vorticity behind the shock.

Equation (2.1) can be re-written in terms of normal and tangential derivatives relative to the shock

$$\omega_b = \frac{\partial A_b}{\partial S} - \left(\frac{\partial B}{\partial N}\right)_b + A_b \frac{\partial \theta}{\partial N} + B\frac{\partial \theta}{\partial S}, \tag{2.2}$$

where A and B are respectively the velocities normal and tangential to the shock, $\partial/\partial N$ is the normal derivative relative to the shock, and θ is the angle of the shock normal to the x-axis. The subscript b has been left off the tangential velocity B since B is continuous across the shock. Thus the vorticity jump across the shock is

$$\delta\omega = \frac{\partial(\,\delta A)}{\partial S} - \,\delta\left(\frac{\partial B}{\partial N}\right) + \,\delta A \frac{\partial \theta}{\partial N}. \tag{2.3}$$

The second and third terms on the right-hand side of (2.3) cannot be found from the Rankine–Hugoniot jump conditions for the velocity. An expression for $\delta(\partial B/\partial N)$ can be found using the two-dimensional gasdynamics equations

$$\rho_t + (u, v)\begin{pmatrix}\rho_x \\ \rho_y\end{pmatrix} + \rho(u_x + v_y) = 0, \tag{2.4}$$

$$\begin{pmatrix}u_t \\ v_t\end{pmatrix} + \begin{pmatrix}u_x & u_y \\ v_x & v_y\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix} + \frac{1}{\rho}\begin{pmatrix}P_x \\ P_y\end{pmatrix} = 0, \tag{2.5}$$

$$P_t + (u, v)\begin{pmatrix}P_x \\ P_y\end{pmatrix} + \gamma P(u_x + v_y) = 0, \tag{2.6}$$

where P is the pressure and γ is the ratio of specific heats, written in terms of tangential and normal derivatives relative to the shock. The third relation is then

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + (A - C)\frac{\partial \theta}{\partial N} - \left(\frac{A - C}{A}\right)\frac{\partial B}{\partial N} - \frac{1}{\rho A}\frac{\partial P}{\partial S} - \frac{1}{A}\left(\frac{\mathrm{d}B}{\mathrm{d}t} + B\frac{\partial B}{\partial S} - AB\frac{\partial \theta}{\partial S}\right) = 0. \tag{2.7}$$

Calculating the jump in equation (2.7) across the shock and rearranging the terms, we find

$$\rho_b A_b \delta \left(\frac{\partial B}{\partial N}\right) = (\delta\rho C - \rho\delta A)\frac{\partial C}{\partial S} + \delta\rho C\left(B\frac{\partial \theta}{\partial S} - \frac{\partial B}{\partial N}\right) - \delta\rho\left(\frac{\partial B}{\partial t} + B\frac{\partial B}{\partial S}\right)$$
$$+\rho(A-C)\delta A\frac{\partial \theta}{\partial N} + \rho(A-C)\frac{\partial(\delta A)}{\partial S} + \delta A\left(\rho\frac{\partial A}{\partial S} + (A-C)\frac{\partial \rho}{\partial S}\right), \quad (2.8)$$

where C is the propagation speed of the shock, the subscript a indicating quantities ahead of the shock has been dropped (henceforth quantities with no subscript are those ahead of the shock) and $\partial(\delta P)/\partial S$ has been eliminated using the Rankine–Hugoniot relation

$$\delta P = -\rho(A-C)\,\delta A, \quad (2.9)$$

which can be differentiated with respect to S to give

$$-\frac{\partial(\delta P)}{\partial S} = \rho(A-C)\frac{\partial(\delta A)}{\partial S} + \left(\rho\frac{\partial A}{\partial S} - \rho\frac{\partial C}{\partial S} + (A-C)\frac{\partial \rho}{\partial S}\right)\delta A. \quad (2.10)$$

Substituting the expression (2.8) for $\delta(\partial B/\partial N)$ into the equation for the vorticity jump (2.3) one obtains the general expression for the vorticity jump across an unsteady shock moving into a non-uniform flow

$$\delta\omega = \frac{1}{\rho(A-C)}\left[\delta\rho\left(\frac{DB}{Dt} - C\frac{\partial C}{\partial S} - CB\frac{\partial \theta}{\partial S}\right) - \delta A\frac{\partial}{\partial S}(\rho(A-C))\right], \quad (2.11)$$

where

$$\frac{DB}{Dt} = \frac{\partial B}{\partial t} + C\frac{\partial B}{\partial N} + B\frac{\partial B}{\partial S}. \quad (2.12)$$

Equation (2.11) can be shown to be equivalent to Hayes' (1957) result (1.4).

The terms in (2.11) can be evaluated using the Rankine–Hugoniot relation for δA

$$\delta A = \frac{\mu}{1+\mu}(C-A), \quad (2.13)$$

and the definitions of A and B

$$A = N_x u + N_y v, \quad (2.14)$$
$$B = N_y u - N_x v, \quad (2.15)$$

where (N_x, N_y) is the unit vector normal to the shock. After some manipulation one finds

$$\delta\omega = \frac{\mu^2}{1+\mu}\frac{\partial C_r}{\partial S} - \frac{\mu}{C_r}\left[\left(\frac{D\boldsymbol{u}}{Dt}\right)_S + \frac{C_r^2}{1+\mu}\frac{1}{\rho}\frac{\partial \rho}{\partial S}\right] + \mu\omega. \quad (2.16)$$

If the flow ahead of the shock is isentropic then $P = a_0^2\rho$ (where a_0 is the stagnation sound speed of the flow) and the term involving density in (2.16) may be replaced using the substitution

$$\frac{\partial \rho}{\partial S} = -\frac{\rho_0}{a_0^2}\left(\frac{D\boldsymbol{u}}{Dt}\right)_S, \quad (2.17)$$

and the vorticity jump becomes

$$\delta\omega = \frac{\mu^2}{1+\mu}\frac{\partial C_r}{\partial S} + \frac{\frac{1}{2}(\gamma-1)\mu^2}{1-\frac{1}{2}(\gamma-1)\mu}\frac{1}{C_r}\left(\frac{D\boldsymbol{u}}{Dt}\right)_S + \mu\omega. \quad (2.18)$$

The first term on the right-hand side of (2.18) represents the vorticity jump due to shock curvature which is directly related to the gradient of shock strength (it does not involve the flow ahead of the shock), the second term represents baroclinic generation of vorticity ($\nabla P \times \nabla \rho$) due to misalignment between pressure and density gradients as the flow passes through the shock, and the last term represents conservation of angular momentum. The identification of the second term as baroclinic is clear from the derivation (it involves products of gradients of density and pressure ahead of the shock), while the third term is the angular momentum created by compression of the flow in the direction normal to the shock front.

Note that the term $\mu\omega$ in (2.18) is not negligible even if the shock is weak; thus there is vorticity generation across a weak shock if the flow ahead of the shock is rotational. If $\mu \partial C_r/\partial S > 1$ then vorticity generation by the curvature term is also not negligible in weak shocks; this is the case near kinks (shock-shocks). These results show that one must be careful when considering vorticity amplification across a weak shock.

It is usually stated that there is no vorticity jump across a weak shock because the entropy jump is $O(\mu^3)$ (e.g. Landau & Lifshitz 1987, p. 436). However, (2.18) clearly shows that the vorticity jump across a weak shock is not zero if the flow ahead of the shock is rotational or if there are large gradients in shock strength. One would expect non-uniformities in the flow ahead of the shock to be important if the flow contains intense local structures such as vortices (this is the case for turbulent flows).

2.2. *Straight shock in irrotational flow*

We now consider the question of whether there is a jump in vorticity across a straight shock if the flow ahead of the shock is irrotational and steady. Separating the rotational and irrotational parts of the flow ahead of the shock, the vorticity jump becomes

$$\delta\omega = \frac{\mu^2}{1+\mu}\frac{\partial C_r}{\partial S} + \frac{\frac{1}{2}(\gamma-1)\mu^2}{1-\frac{1}{2}(\gamma-1)\mu}\frac{1}{C_r}(Ae_{NS} + Be_{SS}) + \left(\mu - \frac{\frac{1}{2}(\gamma-1)\mu^2}{1-\frac{1}{2}(\gamma-1)\mu}\frac{1}{C_r}A\right)\omega, \tag{2.19}$$

where the irrotational rates of strains e_{NS} and e_{SS} are defined in terms of the stream function Ψ by

$$e_{NS} = 2N_xN_y\Psi_{xy} + \tfrac{1}{2}(N_y^2 - N_x^2)(\Psi_{yy} - \Psi_{xx}), \tag{2.20}$$

$$e_{SS} = (N_y^2 - N_x^2)\Psi_{xy} - N_xN_y(\Psi_{yy} - \Psi_{xx}). \tag{2.21}$$

If the shock is straight $\partial C_r/\partial S = 0$, and if the flow ahead of the shock is irrotational $\omega = 0$, then from (2.19) the vorticity jump across a straight shock becomes

$$\delta\omega = \frac{\frac{1}{2}(\gamma-1)\mu^2}{1-\frac{1}{2}(\gamma-1)\mu}\frac{1}{C_r}(ue_{NS} + ve_{SS}), \tag{2.22}$$

which may be non-zero. The vorticity is generated exclusively by baroclinic effects in this case. Thus we have seen that a straight shock may generate vorticity from an initially irrotational flow via tangential strains in the velocity field ahead of the shock.

It is important to bear in mind that the irrotational strain field can deform the shock and thus the shock may not remain straight. However, if the shock is strong and the flow ahead weak the vorticity production due to shock deformation will remain negligible. If, on the other hand, the shock is weak curvature will rapidly become the dominant source of vorticity production across the shock. The main point of this subsection is that in principle shock curvature and rotational flow ahead of the shock are not required to produce vorticity across the shock. This can also be seen in the deposition of vorticity on density interfaces by straight shocks in the Richtmyer–Meshkov instability environment (e.g. Samtaney & Zabusky 1994), although we have used the acoustic approximation to express the vorticity jump entirely in terms of velocity gradients (which is more useful in the context of the shock–turbulence interaction).

2.3. *The shock–turbulence interaction*

The expression for the vorticity jump (2.18) may be put into a form useful for analysing the shock–turbulence interaction by writing it in terms of the Mach number of the shock M_S and the Mach number of the flow ahead of the shock $M_U = (u^2+v^2)^{1/2}/a_0$. M_U is the local equivalent of the turbulence Mach number $M_t = (\overline{M_U^2})^{1/2}$. In the following it will be assumed that $M_U^2 \ll 1$ so that the flow ahead of the shock is only weakly compressible.

If the turbulence is approximately steady the vorticity jump becomes

$$\delta\omega = \frac{\mu^2}{2M_S}\left(\frac{1}{1+\mu}\frac{\partial M_S^2}{\partial S} + \frac{\frac{1}{2}(\gamma-1)}{1-\frac{1}{2}(\gamma-1)\mu}\frac{\partial M_U^2}{\partial S}\right) + \mu\omega + O(M_U^4), \tag{2.23}$$

where the following expression for the variation of sound speed in steady flow has been used:

$$\frac{a}{a_0} = \left(1-\tfrac{1}{2}(\gamma-1)M_U^2\right)^{1/2} = 1-\tfrac{1}{4}(\gamma-1)M_U^2 + O(M_U^4). \tag{2.24}$$

When applied to the shock–turbulence interaction the first two terms on the right-hand side of (2.23) represent vorticity generation by tangential gradients of shock speed 'energy' and local kinetic energy (which is essentially the TKE). The first effect is a result of shock curvature and the second is a baroclinic generation of vorticity.

In his review article on compressible turbulence Lele (1994) states that from direct numerical simulations (DNS) of the weak shock–weak turbulence interaction ($M_t^2 < 0.1(M_s^2-1)$, $(M_s^2-1) < 0.5$) "the net result is that the vorticity components in the plane of the shock increase in proportion to the density ratio across the shock and the normal component remains unchanged". This effect corresponds to the term $\mu\omega$ in (2.23) and thus this is the dominant term in the weak-shock–turbulence interaction. Such a result would be expected from a simple order of magnitude analysis for weak shocks $\mu \ll 1$ which gives the vorticity jump as

$$\delta\omega = \mu\omega + O(M_U^4) + O(\mu^2). \tag{2.25}$$

It is likely, however, that in some localized regions shock focusing occurs (see Kevlahan 1996) and at these places the first two terms in (2.23) will not be negligible, even for weak shocks. Focusing events are more common for $M_t^2 < 0.1(M_s^2-1)$ and thus the first two terms on the right-hand side of (2.23) should be dominant in the strong turbulence–weak shock and strong turbulence–strong shock interactions.

The ratio R of the kinetic energy term to the shock speed 'energy' term is

$$R = \tfrac{1}{2}(\gamma-1)M_S^2\left(\frac{\partial M_U^2/\partial S}{\partial M_S^2/\partial S}\right) \approx \tfrac{1}{2}(\gamma-1)M_U^2 \ll 1, \tag{2.26}$$

assuming that $\partial M_S^2/\partial S = O(M_S^2)$ and $\partial M_U^2/\partial S = O(M_U^2)$. For example, if $\gamma = 1.4$ and $M_U = 0.3$ then $R = 0.018$. Note that if the gradients in TKE are strong then generation of vorticity by tangential gradients of TKE may become more important.

These results suggest that generation of vorticity across a weak shock in turbulence is due primarily to shock curvature and the term $\mu\omega$ (conservation of angular momentum). This is corroborated by the DNS of Kida & Orszag (1990) who find the vorticity production in the entire flow by the TKE term is small, $R \approx 0.128$.

Note that in analysing their numerical results Kida & Orszag (1990) derive Truesdell's expression for the vorticity jump across a steady two-dimensional shock in uniform flow (without reference to Truesdell 1952, Lighthill 1957 or Hayes 1957). Turbulence is not steady or uniform and the eddy shocklets observed by Kida & Orszag must also be unsteady. As is shown in the following section, the gradient in shock strength is proportional to shock curvature only for steady shocks. By using the Truesdell–Lighthill relation Kida & Orszag miss out the generation of vorticity by unsteady processes and by non-uniformities in the flow ahead of the shock. Because they do not include the effect of non-uniformities on the vorticity jump, Kida & Orszag (1990) cannot separate the contribution to vorticity generation by gradients in TKE due to shocks from that due to other flow regions.

2.4. *Steady bow shock*

A curved bow shock forms in front of a blunt object travelling supersonically (see figure 1). If the bow shock is steady it has a constant shape (which we assume is given) and thus $\mathrm{d}x/\mathrm{d}t = C(x, y = 0)$. From the geometry of the shock this means that the increment in arclength $\mathrm{d}S = -C(x,0)\sin\theta\,\mathrm{d}t$ and therefore

$$\mathrm{d}\theta = \left(-\frac{\partial\theta}{\partial S}\right)\mathrm{d}S = \frac{\partial\theta}{\partial S}C(x,0)\sin\theta\,\mathrm{d}t, \tag{2.27}$$

and thus

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = C(x,0)\sin\theta\frac{\partial\theta}{\partial S}, \tag{2.28}$$

where $-\partial\theta/\partial S$ is the curvature of the shock (the shock is convex). Since $\mathrm{d}\theta/\mathrm{d}t = \partial C/\partial S$ this means that the distribution of strength in a steady shock is given by

$$C(x,0)\sin\theta\frac{\partial\theta}{\partial S} = \frac{\partial C}{\partial S} = \frac{\partial C_r}{\partial S} + \frac{\partial A}{\partial S} = \frac{\partial C_r}{\partial S} + e_{NS} + \tfrac{1}{2}\omega - B\frac{\partial\theta}{\partial S}. \tag{2.29}$$

Thus the distribution of shock strength in a steady shock is

$$\frac{\partial C_r}{\partial S} = (B + C(x,0)\sin\theta)\frac{\partial\theta}{\partial S} - e_{NS} - \tfrac{1}{2}\omega. \tag{2.30}$$

Equation (2.30) shows that the variation of strength in a steady bow shock in a flow at rest is determined uniquely by its shape and its strength at maximum curvature.

Substituting (2.30) into the vorticity jump equation (2.18) we find that the vorticity jump across a steady shock moving into a non-uniform flow is

$$\delta\omega = \frac{\mu^2}{1+\mu}\left((B + C(x,0)\sin\theta)\frac{\partial\theta}{\partial S} - e_{NS}\right) + \frac{\frac{1}{2}(\gamma-1)\mu^2}{(1-\frac{1}{2}(\gamma-1)\mu)}\frac{1}{C_r}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_S + \frac{\mu^2+2\mu}{1+\mu}\omega. \tag{2.31}$$

If the flow ahead of the shock is uniform then the vorticity jump becomes simply

$$\delta\omega = \frac{\mu^2}{1+\mu}(B + C(0)\sin\theta)\frac{\partial\theta}{\partial S}, \tag{2.32}$$

which is the vorticity jump predicted by the Truesdell equation (if the tangential velocity in (1.1) is replaced by $B + C(0)\sin\theta$).

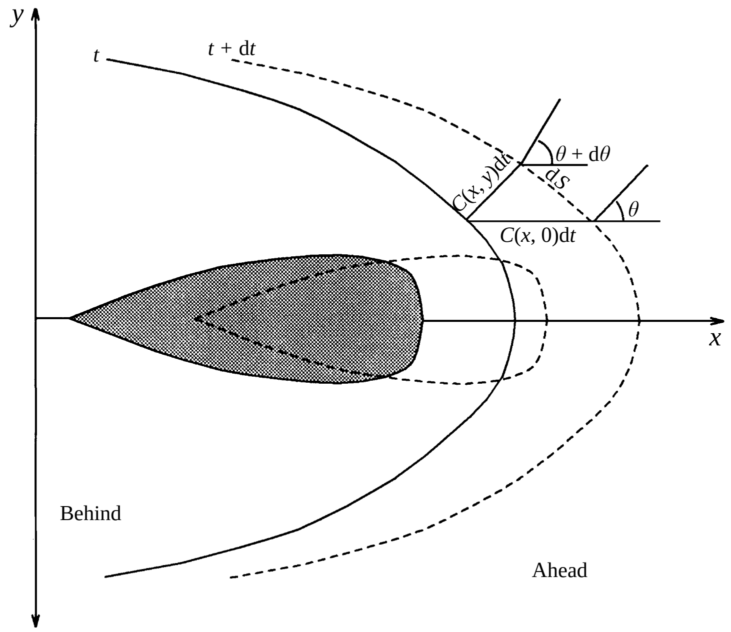

FIGURE 1. The steady bow shock.

Comparing the relation (1.1) derived by Truesdell for steady shocks in a uniform flow with the relation (1.5) derived by Hayes for unsteady shocks in a uniform flow gives the mistaken impression that shock strength is uniform in a steady shock, i.e. $\partial C_r/\partial S = 0$. As proved in this section, shock strength must be *non-uniform* in order to maintain a constant shock shape. This non-uniformity produces the effective relative tangential velocity component present in a steady curved shock and contributes to the vorticity jump.

3. Applications to non-uniform flows ahead of the shock

3.1. *Steady strong shock and weak flow ahead*

In this section we calculate analytically the vorticity jump across a strong shock moving into various non-uniform flows.

Consider an initially steady curved shock entering a region of weak non-uniform flow. We first need to determine the approximations that must be made in order for the deformation of the shock to be negligible. The equation for the motion of a shock surface $x = g(y, t)$ is

$$\frac{\partial g}{\partial t} = \tilde{u} - \tilde{v}\frac{\partial g}{\partial y} + M(y,t)\left[1 + \left(\frac{\partial g}{\partial y}\right)^2\right]^{1/2} = \tilde{u} + \tilde{v}\frac{N_y(y,t)}{N_x(y,t)} + \frac{M(y,t)}{N_x(y,t)}, \qquad (3.1)$$

where $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v})$ is the turbulence velocity ahead of the shock, $M(y,t)$ is the Mach number of the shock as a function of the y-coordinate and time, and the equation has been normalized by the velocity of sound a_0. The change in the position of the shock over a short time Δt may thus be estimated as

$$\Delta g = g(y, \Delta t) - g(y, 0) = \Delta t\frac{\partial g}{\partial t}(y, 0) = \left(\tilde{u} + \tilde{v}\frac{N_y}{N_x} + M_0\right)\Delta t, \qquad (3.2)$$

where $M_0 = M(0,0)$ and we have used the fact that $M(y,0)/N_x(y,0) = M_0$ since the shock is assumed to be initially steady. If the time $\Delta t = O(1/M_0) \ll 1$ then the deformation of the shock due to the turbulence is negligible to first order provided that $\tilde{u}, \tilde{v} \ll 1$ (and $N_y/N_x = O(1)$), i.e. $M_U M_0 = O(1)$. Thus, using (2.23) and (2.30) the vorticity jump across the steady shock becomes

$$\delta\omega = \frac{\mu^2}{1+\mu} M_0 N_y(y,0)\tilde{K}(y,0) + \frac{\frac{1}{2}(\gamma-1)\mu^2}{(1-\frac{1}{2}(\gamma-1)\mu)} \frac{1}{M(y,0)} \frac{\partial(\frac{1}{2}M_U^2)}{\partial S} + \mu\tilde{\omega}, \quad (3.3)$$

where $\tilde{K}(y,0) = \partial\theta/\partial S$ is the curvature of the undeformed shock and lengths have been normalized by the initial minimum radius of curvature of the shock. The orders of the various coefficients in equation (3.3) are

$$\frac{\mu^2}{1+\mu} \approx \frac{4}{\gamma^2-1} = O(1), \quad (3.4)$$

$$\frac{\frac{1}{2}(\gamma-1)\mu^2}{1-\frac{1}{2}(\gamma-1)\mu} \approx \frac{2}{\gamma+1} M_0^2 = O(1/M_U^2), \quad (3.5)$$

$$\mu \approx 5 = O(1). \quad (3.6)$$

Equation (3.5) shows that overall the second term on the right-hand side of (3.3) is $O(M_U)$ and is not negligible. The vorticity jump across a strong curved shock moving into weak non-uniform flow is thus given by (3.3) for short times $t < 1/M_0$.

If the shock is initially parabolic, $x = -1/2y^2$, then the shock strength is determined by

$$\frac{\partial M}{\partial S} = M_0 N_y(y) K(y) = -\frac{M_0 y}{(1+y^2)^2}, \quad (3.7)$$

but $\partial M/\partial S = \mathrm{d}M/\mathrm{d}y(1+y^2)^{-1/2}$, therefore

$$\frac{\mathrm{d}M}{\mathrm{d}y} = -\frac{M_0 y}{(1+y^2)^{3/2}} \quad (3.8)$$

which may be integrated to give

$$M(y) = \frac{M_0}{(1+y^2)^{1/2}} = M_0 N_x(y), \quad y \leqslant (M_0^2-1)^{1/2}, \quad (3.9)$$

as expected.

Thus, the vorticity jump across a steady parabolic shock moving into non-uniform flow is given by

$$\delta\omega = -\frac{\mu^2}{(1+\mu)} \frac{M_0 y}{(1+y^2)^2} + \frac{\frac{1}{2}(\gamma-1)\mu^2}{(1-\frac{1}{2}(\gamma-1)\mu)} \frac{(1+y^2)^{1/2}}{M_0} \frac{\partial(\frac{1}{2}M_U^2)}{\partial S} + \mu\tilde{\omega}, \quad (3.10)$$

where $\mu(y)$ is defined by

$$\mu(y) = \frac{M_0^2 - 1 + y^2}{1 + \frac{1}{2}(\gamma-1)M_0^2 + y^2}. \quad (3.11)$$

The vorticity jumps for the following flows are shown in figure 2: a sinusoidal shear flow

$$\tilde{u}(y) = -M_{U0}\cos y, \quad \tilde{v} = 0, \quad \tilde{\omega}(y) = -M_{U0}\sin y,$$
$$\frac{\partial(\frac{1}{2}M_U^2)}{\partial S} = \tfrac{1}{2}M_{U0}^2 N_x(y)\sin 2y = \tfrac{1}{2}M_{U0}^2(1+y^2)^{-\frac{1}{2}}\sin 2y;$$

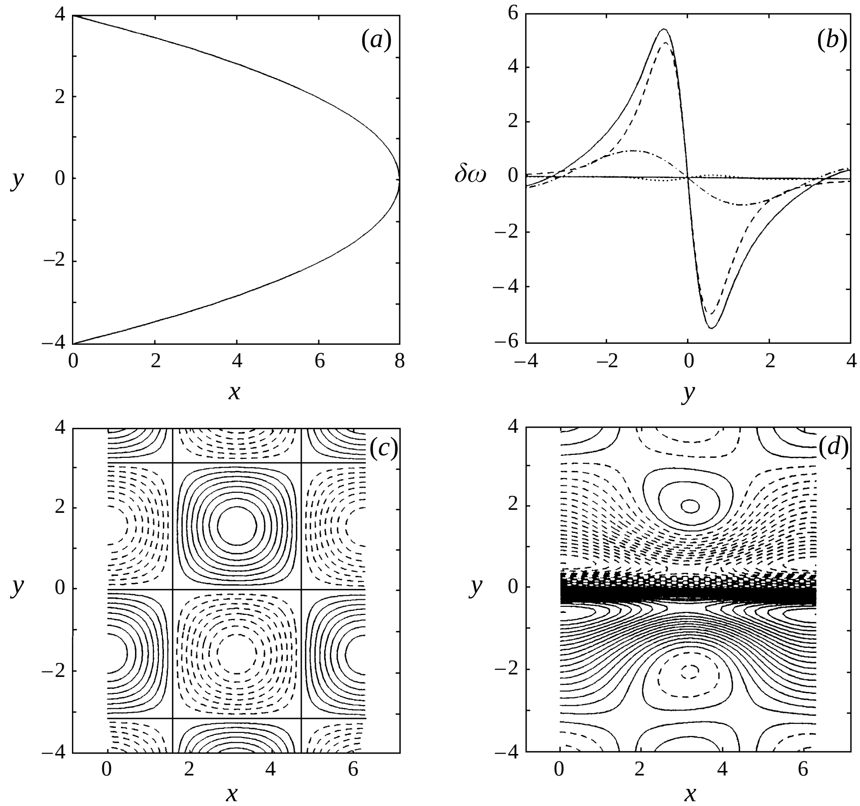

FIGURE 2. The vorticity jump across a strong steady parabolic shock for two non-uniform flows ahead of the shock. Shock strength $M_0 = 5$, amplitude of flow ahead of the shock $M_{U0} = 0.3$. (*a*) Shock shape. (*b*) Sinusoidal shear flow: —, net; - - -, curvature term; ···, baroclinic term; — · —, angular momentum term. (*c*) Equally spaced contours of initial vorticity in the vortex array. (*d*) Equally spaced contours of vorticity jump in the vortex array. Note that the curvature term depends only on shock shape and thus has a *y*-dependence only, all *x*-dependence in the vorticity jump comes from the terms ahead of the shock.

a vortex array (perpendicular sinusoidal modes)

$$\tilde{u}(x, y) = -M_{U0}\cos(x)\cos(y), \quad \tilde{v}(x, y) = -M_{U0}\sin(x)\sin(y),$$
$$\tilde{\omega}(x, y) = -2M_{U0}\cos(x)\sin(y),$$
$$\frac{\partial(\frac{1}{2}M_U^2)}{\partial S} = \tfrac{1}{2}M_{U0}^2(-N_y(y)\sin(2x)\cos(2y) + N_x(y)\cos(2x)\sin(2y))$$
$$= \tfrac{1}{2}M_{U0}^2(-y\sin(2x)\cos(2y) + \cos(2x)\sin(2y))(1 + y^2)^{-1/2}.$$

Note that the vorticity jump in the sinusoidal shear flow takes the form of a dipole with peaks offset symmetrically about the vertex of the parabola $y = 0$; this is the effect of the curvature term. The vorticity jump changes sign towards the edge of the parabola where the second term $\mu\omega$ due to the non-uniform flow ahead of the shock becomes dominant. The vorticity jump in the vortex array flow is essentially homogeneous in x near $y = 0$, but inhomogeneous in x for larger y where the terms ahead of the shock in the vorticity equation become significant relative to the curvature term.

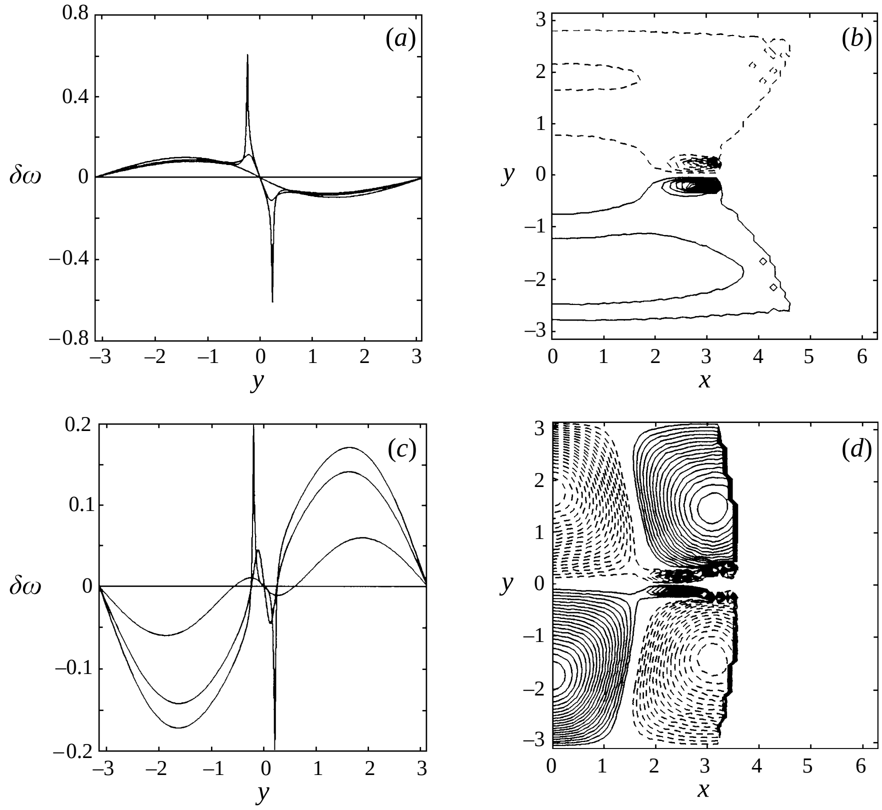

FIGURE 3. The vorticity jump across an initially straight unsteady weak shock for various non-uniform flows ahead of the shock. The flow behind the shock is assumed to be frozen. Shock strength $M_0 = 1.2$, amplitude of flow ahead of the shock $M_{U0} = 0.3$. (*a*) Sinusoidal shear flow, jump across the shock at times $t = 0.15, 0.25, 0.30$ (increasing amplitude). (*b*) Sinusoidal shear flow, contours of vorticity jump. (*c*) Vortex array, jump across the shock at times $t = 0.15, 0.20, 0.24$ (increasing amplitude). (*d*) Vortex array, equally spaced contours of vorticity jump.

3.2. *Unsteady weak shock and weak flow ahead of the shock*

In this section we use the weak shock propagation equations developed in Kevlahan (1996) to calculate the vorticity behind an unsteady weak shock moving into a non-uniform flow. The vorticity jump for this situation is given by

$$\delta\omega = \tfrac{1}{4}(\gamma + 1)\frac{\mu^2}{(1+\mu)}\frac{\partial \mu}{\partial S} + \mu\omega, \tag{3.12}$$

where the first term on the right-hand side has been retained because kinks (and associated discontinuities in $\mu(S)$) may develop in the weak shock. The shock propagation equations are solved numerically using the method described in Kevlahan (1996) for a sinusoidal shear flow, and a vortex array. It is assumed that the vorticity remains approximately frozen behind the shock (if the flow is inviscid the vorticity is merely transported along streamlines). The results are shown in figure 3. Note the extremely strong vorticity associated with kinks in the shock and the fact that the term $\mu\omega$ dominates the vorticity jump except near the kinks. Figure 3 shows how the vorticity pattern is also deformed outside the kink region.

Samtaney & Zabusky (1994) considered the related two-dimensional problem of vorticity deposition by a shock wave on a density inhomogeneity (a straight shock wave moves through a density discontinuity). In this case the vorticity production is entirely baroclinic. They analyse the vorticity deposition phase, and use a DNS to calculate the resulting baroclinic evolution of the flow. The DNS shows that vorticity deposited on a curved density interface later coalesces to form a mushroom-shaped coherent structure. One might also expect the intense filamentary vorticity produced by the kinks in figure 3 to coalesce into coherent structures (e.g. through a Kelvin–Helmholtz instability).

The study of Samtaney & Zabusky (1994) is complementary to the study presented here since in their case the baroclinic effects dominate and the shock curvature and conservation of angular momentum are negligible, while in our case (and in most shock–turbulence interactions) the opposite is true. It would be interesting to consider a situation where shock curvature and baroclinic effects are equally important.

4. Conclusions

The vorticity jump across an unsteady shock of arbitrary strength moving into a non-uniform flow has been investigated in detail. The exact equation for the vorticity jump has been derived in a form useful for evaluating the different contributions to vorticity production and evaluated under several different conditions. The vorticity jump across steady strong shocks in a variety of non-uniform flows has been calculated analytically, and the vorticity jump across unsteady weak shocks in the same flows has been calculated numerically.

When a weak shock focuses (develops kinks and shock-shocks) intense vorticity is generated across the kinks. This vorticity generation mechanism should be important in the interaction of weak shocks and relatively strong turbulence ($M_t^2 \geqslant 0.1(M_S^2-1)$). Enhanced vorticity generation at regions of high curvature may account for the underestimation of vorticity generation by the linear interaction analysis, which does not include shock evolution (Lee *et al.* 1993). The increase in the variance of vorticity behind a curved or focusing shock may also account for a large portion of the generation of turbulent kinetic energy behind a shock.

By including terms due to the non-uniform flow ahead of the shock it is shown that the vorticity jump may be non-zero even for weak shocks and straight shocks. The vorticity jump across a straight shock may be non-zero even if the flow ahead of the shock is irrotational.

The ratio of vorticity production by shock curvature to vorticity production by baroclinic effects is found to be $O(\frac{1}{2}(\gamma-1)M_U^2)$ which is very small if the flow ahead of the shock is only weakly compressible, $M_U^2 \ll 1$. If, however, the tangential gradient along the shock of M_U^2 is large then baroclinic production is significant; this is the case in turbulent flows with large gradients of turbulent kinetic energy $\frac{1}{2}M_U^2$.

It is important to note here that we have considered the instantaneous vorticity jump across a shock where the shape of the shock and the distribution of its strength as well as the flow ahead are specified. We have not considered the subsequent evolution when separate baroclinic effects drive the flow behind the shock. A complementary approach, that of calculating the entire flow field for simple compressible flows ahead of the shock, was used by Azara & Emanuel (1988) who found the two-dimensional or axisymmetric compressible flow for four simple cases. They calculated the conditions for zero vorticity behind a curved shock (baroclinic vorticity production exactly cancels the vorticity produced by the curvature of the shock) and found that irrotational flow behind a curved shock is impossible if the flow ahead of the shock is rotational.

Complementary work by Mahesh *et al.* (1995) using DNS to study the interaction between a weak shock and compressible upstream turbulence has shown that significant vorticity can also be generated across a weak shock when the upstream flow contains acoustic waves. Recent work by Mahesh, Lele & Moin (1997) has focused on the role of entropy fluctuations in the baroclinic generation of vorticity across weak shocks. These studies add to the results obtained here showing the importance of vorticity generation mechanisms in the weak shock–turbulence interaction.

The importance of the contribution to the vorticity jump by non-uniformities in the flow ahead of the shock has not been recognized in the past. Local non-uniformities are especially important in the case of turbulence where they take the form of flow structures such as intense vortices.

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